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NUMERICAL SOLUTION OF FOURTH ORDER BOUNDARY VALUE PROBLEMS BY PETROV-GALERKIN METHOD WITH CUBIC B-SPLINES AS BASIS FUNCTIONS AND QUARTIC B-SPLINES AS WEIGHT FUNCTIONS

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ABSTRACT

We have taken basis functions with cubic B-splines and weight functions with quartic B-splines in Petrov-Galerkin method to solve a boundary value problem of fourth order. In this method, the cubic B-splines and quartic B-splines are redefined into new sets of functions which contain the equal number of functions. To test the accuracy and efficiency of the method proposed, the numerical results obtained are presented in the form of absolute errors and found that the obtained results are giving a little absolute error.

KEYWORDS: Basis Functions, Boundary Value Problem, B-splines, Petrov-Galerkin Method, Weight Functions

INTRODUCTION

Consider a general linear boundary value problem of fourth order

$$p_0(t)v^{(4)}(t)+p_1(t)v'''(t)+p_2(t)v''(t)+p_3(t)v'(t)+p_4(t)v(t)=q(t), \quad a<t<b \tag{1}$$

Subject to the boundary conditions

$$v(a)=A_0, \quad v(b)=C_0, \quad v'(a)=A_1, \quad v'(b)=C_1 \tag{2a}$$

or

$$v(a)=A_0, \quad v(b)=C_0, \quad v''(a)=A_1, \quad v''(b)=C_1 \tag{2b}$$

or

$$v(a)=A_0, \quad v(b)=C_0, \quad v'(a)+\sigma_1 v(a)=A_1, \quad v'(b)+\sigma_2 v(b)=C_1 \tag{2c}$$

where A_0, C_0, A_1, C_1, σ_1 and σ_2 are real constants and $p_0(t)$, $p_1(t)$, $p_2(t)$, $p_3(t)$, $p_4(t)$ and $q(t)$ are continuous functions defined in [a, b].

The fourth order boundary value problems arise in the areas of fluid mechanics, elasticity and quantum mechanics and in some allied areas of science and engineering. The existence and uniqueness for the solution of these problems are described in Agarwal [1]. The exact solutions of such type of boundary value problems are rarely available. Various numerical methods have been developed by many researchers [2-21]. Till now, the researchers did not use cubic B-splines as basis functions and quartic B-splines as weight functions in the Petrov-Galerkin method to solve fourth order boundary value problems of the type (1)-(2). That's why we want to use the above method as our proposed method.

The subsequent sections are dealt with the justification of using Petrov-Galerkin method, a description of using proposed method with the types of boundary conditions (2), the procedure of solving the nodal parameters and the application of the proposed method on solving several examples of linear and nonlinear boundary value problems. By using quasilinearization technique [22], a nonlinear problem can be converted into a sequence of linear problems and the limit of solutions of these generated linear problems is the solution of the nonlinear problem. The conclusions are presented in the last section.

JUSTIFICATION OF USING PROPOSED METHOD

The approximate solution in Finite Element Method (FEM) can be obtained as a linear combination of basis functions which constitute a basis for the approximation space under consideration. Petrov-Galerkin method is one of the variational methods involved in FEM. The residual of approximation is made orthogonal to the weight functions in Petrov-Galerkin method. Regardless of properties of the differential operator defined in the given differential equation, a weak form of approximation solution for the differential equation exists and is unique under appropriate conditions [23, 24]. Further, if we pay sufficient attention to the boundary conditions [25], the weak solution tends to an exact solution of the differential equation. This means that the basis functions should become zero on the boundary where the essential (Dirichlet) type of boundary conditions are defined. Also the basis functions and the weight functions are equal in number.

DESCRIPTION OF THE PROPOSED METHOD

Cubic B-splines and Quartic B-splines

The cubic B-splines and quartic B-splines are described in [26-28]. Space variable domain [a, b] is divided into spaced knots (which need not be spaced evenly) given by the partition $a = t_0 < t_1 < ... < t_{n-1} < t_n = b$. Six additional knots t_{-3}, t_{-2}, t_{-1}, t_{n+1}, t_{n+2} and t_{n+3} are introduced which satisfy the relation

$t_{-3}<t_{-2}<t_{-1}<t_0$ and $t_n<t_{n+1}<t_{n+2}<t_{n+3}$.

Now the cubic B-splines $S_i(t)$'s are defined by

$$S_i(t) = \begin{cases} \sum\limits_{r=i-2}^{i+2} \dfrac{(t_r - t)_+^3}{\pi'(t_r)}, & t \in [t_{i-2}, t_{i+2}] \\ 0, & otherwise \end{cases}$$

Where

And $\pi(t) = \prod\limits_{r=i-2}^{i+2} (t - t_r)$

Where $\{S_{-1}(t), S_0(t), S_1(t), \ldots, S_{n-1}(t), S_n(t), S_{n+1}(t)\}$ forms a basis for the space $S_3(\pi)$ of cubic polynomial splines. Schoenberg [28] has shown that cubic B-splines are the unique nonzero splines of smallest compact support with the knots at $t_{-3}<t_{-2}<t_{-1}<t_0<t_1<\ldots<t_{n-1}<t_n<t_{n+1}<t_{n+2}<t_{n+3}$.

In the same way, the quartic B-splines $R_i(t)$'s are defined by

$$R_i(t)=\begin{cases}\sum_{r=i-2}^{i+3}\dfrac{(t_r-t)_+^4}{\pi'(t_r)}, & t\in[t_{i-2},\, t_{i+3}]\\ 0, & otherwise\end{cases}$$

Where $$(t_r-t)_+^4=\begin{cases}(t_r-t)^4, & if\ t_r\ge t\\ 0, & if\ t_r\le t\end{cases}$$

And $$\pi(t)=\prod_{r=i-2}^{i+3}(t-t_r)$$

Where $\{R_{-2}(t), R_{-1}(t), R_0(t), R_1(t),\ldots, R_{n-1}(t), R_n(t), R_{n+1}(t)\}$ forms a basis for the space $S_4(\pi)$ of quartic polynomial splines by introducing two more additional knots t_{-4}, t_{n+4} to the already existing knots t_{-3} to t_{n+3}. Schoenberg [28] has shown that quartic B-splines are the unique nonzero splines of smallest compact support with the knots at $t_{-4}<t_{-3}<t_{-2}<t_{-1}<t_0<t_1<\ldots<t_{n-1}<t_n<t_{n+1}<t_{n+2}<t_{n+3}<t_{n+4}$.

We define the approximation for $v(t)$ as

$$v(t)=\sum_{j=-1}^{n+1}\alpha_j S_j(t) \tag{3}$$

Where α_j's are the nodal parameters to be determined and $S_j(t)$'s are the cubic B-spline basis functions. In Petrov-Galerkin method, the basis functions should be zero on the boundary where the essential type of boundary conditions are prescribed. In the set of cubic B-splines $\{S_{-1}(t), S_0(t), S_1(t),\ldots, S_{n-1}(t), S_n(t), S_{n+1}(t)\}$, the basis functions $S_{-1}(t), S_0(t), S_1(t)$ do not become zero on the left boundary and $S_{n-1}(t), S_n(t)$ and $S_{n+1}(t)$ do not become zero on the right boundary. So, it is necessary to redefine the basis functions into a new set of basis functions which become zero on the boundary.

Applying the essential boundary conditions of (2), we get the approximate solution $v(t)$ at the boundary points as

$$A_0=v(a)=v(t_0)=\alpha_{-1}S_{-1}(t_0)+\alpha_0S_0(t_0)+\alpha_1S_1(t_0) \tag{4}$$

$$C_0=v(b)=v(t_n)=\alpha_{n-1}S_{n-1}(t_n)+\alpha_nS_n(t_n)+\alpha_{n+1}S_{n+1}(t_n) \tag{5}$$

Eliminating α_{-1} and α_{n+1} from the equations (3), (4) and (5), we get

$$v(t)=w(t)+\sum_{j=0}^{n}\alpha_j\tilde{B}_j(t) \tag{6}$$

Where $$w(t)=\frac{A_0}{S_{-1}(t_0)}S_{-1}(t)+\frac{C_0}{S_{n+1}(t_n)}S_{n+1}(t) \tag{7}$$

$$\tilde{B}_j(t)=\begin{cases} S_j(t)-\dfrac{S_j(t_0)}{S_{-1}(t_0)}S_{-1}(t), & j=0,1 \\ S_j(t), & j=2,3,...,n-2 \\ S_j(t)-\dfrac{S_j(t_n)}{S_{n+1}(t_n)}S_{n+1}(t), & j=n-1,n \end{cases} \tag{8}$$

$\{\tilde{B}_j(t),\, j=0,1,...,n\}$ are the new set of basis functions for the approximation *v(t)*. Here *w(t)* takes care of given set of essential boundary conditions and $\tilde{B}_j(t)$'s are vanishing at the boundary. In the proposed method, the new set of basis functions and weight functions should be equal in number. Here the number of basis functions in the approximation for *v(t)* in (6) is n + 1 and the number of weight functions is n+4. So, it is necessary to redefine the weight functions into a new set of weight functions which are equal in number of the basis functions.

Let us write the approximation for *u(t)* as

$$u(t)=\sum_{j=-2}^{n+1}\beta_j R_j(t) \tag{9}$$

Where $R_j(t)$'s are the quartic B-splines.

Method with Boundary Conditions (2a)

Let us assume that approximation *u(t)*, given by (9), satisfies the conditions

$$u(a)=0,\, u(b)=0,\, u'(a)=0 \tag{10}$$

Using (9) and (10), we get the approximate solution for *u(t)* at the boundary points as

$$u(a)=u(t_0)=\sum_{j=-2}^{1}\beta_j R_j(t_0)=0 \tag{11}$$

$$u(b)=u(t_n)=\sum_{j=n-2}^{n+1}\beta_j R_j(t_n)=0 \tag{12}$$

$$u'(a)=u'(t_0)=\sum_{j=-2}^{1}\beta_j R'_j(t_0)=0 \tag{13}$$

Eliminating β_{-2}, β_{-1} and β_{n+1} from the equations (9) and (11) to (13), we get the approximation for *u(t)* as

$$u(t)=\sum_{j=0}^{n}\beta_j T_j(t) \tag{14}$$

where

$$T_j(t)=\begin{cases} P_j(t)-\dfrac{P'_j(t_0)}{P'_{-1}(t_0)}P_{-1}(t), & j=0,1 \\ P_j(t), & j=2,3,4,...,n \end{cases} \tag{15}$$

$$P_j(t)=\begin{cases} R_j(t)-\dfrac{R_j(t_0)}{R_{-2}(t_0)}R_{-2}(t), & j=-1,0,1 \\ R_j(t), & j=2,3,4,...,n-3 \\ R_j(t)-\dfrac{R_j(t_n)}{R_{n+1}(t_n)}R_{n+1}(t), & j=n-2,n-1,n \end{cases} \tag{16}$$

Now the new set of weight functions for the approximation *u(t)* is $\{T_j(t),\, j=0,1,...,n\}$. Here $T_j(t_0)=T_j(t_n)=T_j'(t_0)=0$ for all j.

Applying the proposed method to (1) with the new set of basis functions $\{\tilde{B}_j(t),\, j=0,1,...,n\}$ and with the new set of weight functions $\{T_j(t),\, j=0,1,...,n\}$ defined in (15), we get

$$\int_{t_0}^{t_n}\left[p_0(t)v^{(4)}(t)+p_1(t)v'''(t)+p_2(t)v''(t)+p_3(t)v'(t)+p_4(t)v(t)\right]T_i(t)\,dt=\int_{t_0}^{t_n}q(t)T_i(t)dt \quad \text{for } i=0,1,2,\ldots,n. \tag{17}$$

Integrating by parts the first two terms on the left hand side of (17) and after applying the boundary conditions mentioned in (2a), we get

$$\begin{aligned}\int_{t_0}^{t_n}p_0(t)T_i(t)v^{(4)}(t)dt &= \frac{d^2}{dt^2}\left[p_0(t)T_i(t)\right]_{t_n}C_1-\frac{d^2}{dt^2}\left[p_0(t)T_i(t)\right]_{t_0}A_1 \\ &-\int_{t_0}^{t_n}\frac{d^3}{dt^3}\left[p_0(t)T_i(t)\right]v'(t)dt-\frac{d}{dt}\left[p_0(t)T_i(t)\right]_{t_n}v''(t_n)\end{aligned} \tag{18}$$

$$\int_{t_0}^{t_n}p_1(t)T_i(t)v'''(t)dt=\int_{t_0}^{t_n}\frac{d^2}{dt^2}\left[p_1(t)T_i(t)\right]v'(t)dt-\frac{d}{dt}\left[p_1(t)T_i(t)\right]_{t_n}C_1 \tag{19}$$

Using (18), (19) and (6) in (17) and after rearrangement, we get a system of equations in the matrix form as

$$\boldsymbol{K\alpha}=\boldsymbol{f} \tag{20}$$

where $\boldsymbol{K}=[k_{ij}]$;

$$\begin{aligned}k_{ij}&=\int_{t_0}^{t_n}\{[-\frac{d^3}{dt^3}[p_0(t)T_i(t)]+\frac{d^2}{dt^2}[p_1(t)T_i(t)]+p_3(t)T_i(t)]\tilde{B}_j'(t) \\ &+p_2(t)T_i(t)\tilde{B}_j''(t)+p_4(t)T_i(t)\tilde{B}_j(t)\}dt-\frac{d}{dt}\left[p_0(t)T_i(t)\right]_{t_n}\tilde{B}_j''(t_n)\end{aligned} \quad \text{for } i=0,1,2,\ldots,n;\ j=0,1,2,\ldots,n. \tag{21}$$

$\boldsymbol{f}=[f_i]$;

$$f_i = \int_{t_0}^{t_n} \{q(t)T_i(t) + [\frac{d^3}{dt^3}[p_0(t)T_i(t)] - \frac{d^2}{dt^2}[p_1(t)T_i(t)] - p_3(t)T_i(t)]w'(t)$$
$$-p_2(t)T_i(t)w''(t) - p_4(t)T_i(t)w(t)\}dt - \frac{d^2}{dt^2}[p_0(t)T_i(t)]_{t_n} C_1 \qquad \text{for i=0, 1, 2,..., n} \qquad (22)$$
$$+\frac{d^2}{dt^2}[p_0(t)T_i(t)]_{t_0} A_1 + \frac{d}{dt}\left[p_0(t)T_i(t)\right]_{t_n} w''(t_n) + \frac{d}{dt}\left[p_1(t)T_i(t)\right]_{t_n} C_1$$

and $\alpha = [\alpha_0\ \alpha_1 \ldots \alpha_n]^T$.

Method with Boundary Conditions (2b)

Let us assume that the approximation *u(t)*, given by (9), satisfies the conditions

$$u(a) = 0, u(b) = 0, u''(a) = 0 \qquad (23)$$

Using (9) and (23), we get the approximate solution for *u(t)* at the boundary points as

$$u(a) = u(t_0) = \sum_{j=-2}^{1} \beta_j R_j(t_0) = 0 \qquad (24)$$

$$u(b) = u(t_n) = \sum_{j=n-2}^{n+1} \beta_j R_j(t_n) = 0 \qquad (25)$$

$$u''(a) = u''(t_0) = \sum_{j=-2}^{1} \beta_j R_j''(t_0) = 0 \qquad (26)$$

Eliminating β_{-2}, β_{-1} and β_{n+1} from the equations (9) and (24) to (26), we get the approximation for *u(t)* as

$$u(t) = \sum_{j=0}^{n} \beta_j T_j(t) \qquad (27)$$

where

$$T_j(t) = \begin{cases} P_j(t) - \dfrac{P_j''(t_0)}{P_{-1}''(t_0)} P_{-1}(t), & j = 0,1 \\ P_j(t), & j = 2,3,4,\ldots,n \end{cases} \qquad (28)$$

And $\{P_j(t),\ j = -1, 0, 1, \ldots, n-1, n\}$ are defined in (16).

Now $\{T_j(t),\ j = 0,1,\ldots,n\}$ are the new set of weight functions for the approximation *u(t)*. Here $T_j(t_0) = T_j(t_n) = T_j''(t_0) = 0$ for all j.

Applying the proposed method to (1) with the new set of basis functions $\{\tilde{B}_j(t),\ j = 0,1,\ldots,n\}$ and with the new set of weight functions $\{T_j(t),\ j = 0,1,\ldots,n\}$ defined in (28), we get

$$\int_{t_0}^{t_n}\left[p_0(t)v^{(4)}(t)+p_1(t)v'''(t)+p_2(t)v''(t)+p_3(t)v'(t)+p_4(t)v(t)\right]T_i(t)\,dt=\int_{t_0}^{t_n}q(t)T_i(t)dt \quad \text{for } i=0,1,2,\ldots,n. \tag{29}$$

Integrating by parts the first two terms on the left hand side of (29) and after applying the boundary conditions prescribed in (2b), we get

$$\int_{t_0}^{t_n}p_0(t)v^{(4)}(t)T_i(t)\,dt=-\left[\frac{d}{dt}\left(p_0(t)T_i(t)\right)\right]_{t_n}C_1+\left[\frac{d}{dt}\left(p_0(t)T_i(t)\right)\right]_{t_0}A_1+\int_{t_0}^{t_n}\frac{d^2}{dt^2}(p_0(t)T_i(t))\,v''(t)\,dt \tag{30}$$

$$\int_{t_0}^{t_n}p_1(t)T_i(t)v'''(t)dt=-\int_{t_0}^{t_n}\frac{d}{dt}\left[p_1(t)T_i(t)\right]v''(t)dt \tag{31}$$

Using (30), (31) and (6) in (29) and after rearrangement, we get a system of equations in the matrix form as

$$\boldsymbol{K\alpha}=\boldsymbol{f} \tag{32}$$

where $\boldsymbol{K}=[k_{ij}]$;

$$k_{ij}=\int_{t_0}^{t_n}\{[\frac{d^2}{dt^2}[p_0(t)T_i(t)]-\frac{d}{dt}[p_1(t)T_i(t)]+p_2(t)T_i(t)]\widetilde{B}_j''(t) \qquad \text{for } i=0,1,2,\ldots,n;\ j=0,1,2,\ldots,n. \tag{33}$$
$$+p_3(t)T_i(t)\widetilde{B}_j'(t)+p_4(t)T_i(t)\widetilde{B}_j(t)\}dt$$

$\boldsymbol{f}=[f_i]$;

$$f_i=\int_{t_0}^{t_n}\{q(t)T_i(t)+[-\frac{d^2}{dt^2}[p_0(t)T_i(t)]+\frac{d}{dt}[p_1(t)T_i(t)]-p_2(t)T_i(t)]w''(t)$$
$$-p_3(t)T_i(t)w'(t)-p_4(t)T_i(t)w(t)\}dt+\frac{d}{dt}[p_0(t)T_i(t)]_{t_n}C_1-\frac{d}{dt}[a_0(t)T_i(t)]_{t_0}A_1 \tag{34}$$

for i=0, 1, 2, ..., n

and $\alpha=[\alpha_0\,\alpha_1\ldots\alpha_n]^T$.

Method with Boundary Conditions (2c)

Let us assume that the approximation *u(t)*, given by (9), satisfies the conditions

$$u(a)=0,u(b)=0,u'(a)=0 \tag{35}$$

Let us proceed as in the case of method with boundary conditions (2a). $\{T_j(t),\ j=0,1,\ldots,n\}$, as defined in (15), are the new set of weight functions for the approximation *u(t)*.

Applying the Petrov-Galerkin method to (1) with the new set of basis functions $\{\tilde{B}_j(t),\ j=0,1,\ldots,n\}$ and with the new set of weight functions $\{T_j(t),\ j=0,1,\ldots,n\}$ defined in (15), we get

$$\int_{t_0}^{t_n}\left[p_0(t)v^{(4)}(t)+p_1(t)v'''(t)+p_2(t)v''(t)+p_3(t)v'(t)+p_4(t)v(t)\right]T_i(t)\,dt=\int_{t_0}^{t_n}q(t)T_i(t)dt \quad \text{for i=0,1,2,…,n.} \tag{36}$$

Integrating by parts the first two terms on the left hand side of (36) and after applying the boundary conditions prescribed in (2c), we get

$$\begin{aligned}\int_{t_0}^{t_n}p_0(t)T_i(t)v^{(4)}(t)dt &= \frac{d^2}{dt^2}\left[p_0(t)T_i(t)\right]_{t_n}\left(C_1-\sigma_2C_0\right)-\frac{d^2}{dt^2}\left[p_0(t)T_i(t)\right]_{t_0}\left(A_1-\sigma_1A_0\right)\\ &-\int_{t_0}^{t_n}\frac{d^3}{dt^3}\left[p_0(t)T_i(t)\right]v'(t)dt-\frac{d}{dt}\left[p_0(t)T_i(t)\right]_{t_n}v''(t_n)\end{aligned} \tag{37}$$

$$\int_{t_0}^{t_n}p_1(t)T_i(t)v'''(t)dt=\int_{t_0}^{t_n}\frac{d^2}{dt^2}\left[p_1(t)T_i(t)\right]v'(t)dt-\frac{d}{dt}\left[p_1(t)T_i(t)\right]_{t_n}\left(C_1-\sigma_2C_0\right) \tag{38}$$

Using (37), (38) and (6) in (36) and after rearrangement, we get a system of equations in the matrix form as

$$\boldsymbol{K\alpha = f} \tag{39}$$

where $\boldsymbol{K} = [k_{ij}]$;

$$\begin{aligned}k_{ij} &= \int_{t_0}^{t_n}\{[-\frac{d^3}{dt^3}[p_0(t)T_i(t)]+\frac{d^2}{dt^2}[p_1(t)T_i(t)]+p_3(t)T_i(t)]\widetilde{B}_j{}'(t)\\ &+p_2(t)T_i(t)\widetilde{B}_j{}''(t)+p_4(t)T_i(t)\widetilde{B}_j(t)\}dt-\frac{d}{dt}\left[p_0(t)T_i(t)\right]_{t_n}\widetilde{B}_j{}''(t_n)\end{aligned} \tag{40}$$

for i=0,1,2,…, n; j=0,1,2,…, n.

$\boldsymbol{f} = [f_i]$;

$$\begin{aligned}f_i &= \int_{t_0}^{t_n}\{q(t)T_i(t)+[\frac{d^3}{dt^3}[p_0(t)T_i(t)]-\frac{d^2}{dt^2}[p_1(t)T_i(t)]-p_3(t)T_i(t)]w'(t)-p_2(t)T_i(t)w''(t)\\ &-p_4(t)T_i(t)w(t)\}dt-\frac{d^2}{dt^2}[p_0(t)T_i(t)]_{t_n}\left(C_1-\sigma_2C_0\right)+\frac{d^2}{dt^2}[p_0(t)T_i(t)]_{t_0}\left(A_1-\sigma_1A_0\right)\\ &+\frac{d}{dt}[p_0(t)T_i(t)]_{t_n}w''(t_n)+\frac{d}{dt}[p_0(t)T_i(t)]_{t_n}\left(C_1-\sigma_2C_0\right).\end{aligned} \tag{41}$$

for i=0, 1, ..., n

and $\alpha=[\alpha_0\,\alpha_1\ldots\alpha_n]^T$.

PROCEDURE OF SOLVING THE NODAL PARAMETERS

A general element in the matrix $\boldsymbol{K}$ is given by $\sum_{m=0}^{n-1}I_m$, where $I_m=\int_{t_m}^{t_{m+1}}u_i(t)r_j(t)M(t)dt$, $r_j(t)$ are the cubic B-spline basis functions or their derivatives and $u_i(t)$ are the quartic B-spline weight functions or their derivatives. Here

$I_m = 0$ if $(t_{i-2}, t_{i+3}) \cap (t_{j-2}, t_{j+2}) \cap (t_m, t_{m+1}) = \varnothing$. For the evaluation of each I_m, we have used 4-point Gauss-Legendre quadrature formula. Due to this, the stiffness matrix ***K*** is a eight diagonal band matrix. Solving the system ***Kα = f*** by using the band matrix solution package, we get the nodal parameter vector ***α***. We have used the FORTRAN-90 code to solve the boundary value problems (1) - (2) by the proposed method.

NUMERICAL EXAMPLES

To test the accuracy and efficiency of the developed method, we solved five linear and four nonlinear fourth order boundary value problems. The obtained numerical results for each problem are presented in tabular forms.

Example 1

Consider the following linear boundary value problem

$$v^{(4)} + 4v = 1, \qquad -1 \le t \le 1 \tag{42}$$

Subject to $v(-1) = v(1) = 0$, $v'(-1) = \dfrac{\sinh 2 - \sin 2}{4(\cosh 2 + \cos 2)}$, $v'(1) = \dfrac{\sin 2 - \sinh 2}{4(\cosh 2 + \cos 2)}$.

The exact solution for the above problem is

$$v = .25\left[1 - 2\frac{\sinh 1 \sin 1 \sinh t \sin t + \cosh 1 \cos 1 \cosh t \cos t}{(\cos 2 + \cosh 2)}\right].$$

Dividing the domain [-1, 1] into 10 equal subintervals, the numerical results obtained for this problem are presented in Table 1.

Table 1: Numerical Results for Example 1

t	Absolute Error by Proposed Method
-0.8	8.095056E-06
-0.6	5.215406E-06
-0.4	7.696450E-06
-0.2	5.871058E-06
0.0	7.390976E-06
0.2	5.863607E-06
0.4	7.711351E-06
0.6	5.275011E-06
0.8	7.953495E-06

Example 2

Consider the following linear boundary value problem

$$v^{(4)} + tv = -(8 + 7t + t^3)e^t, \qquad 0 < t < 1 \tag{43}$$

Subject to $v(0) = v(1) = 0,\ v'(0) = 1,\ v'(1) = -e.$

The exact solution for the above problem is $v = t(1-t)e^t$.

Dividing the domain [0, 1] into 10 equal subintervals, the numerical results obtained for this problem are presented in Table 2.

Example 3

Consider the following linear boundary value problem

$$v^{(4)} - v'' - v = e^{t}(t-3), \qquad 0 < t < 1 \tag{44}$$

Subject to $v(0)=1,\ v(1)=0,\ v'(0)=0,\ v'(1)=-e.$

The exact solution for the above problem is $v = e^{t}(1-t)$.

Dividing the domain [0, 1] into 10 equal subintervals, the numerical results obtained for this problem are presented in Table 3.

Table 2: Numerical Results for Example 2

t	Absolute Error by Proposed Method
0.1	1.766533E-05
0.2	1.606345E-05
0.3	2.044439E-05
0.4	1.522899E-05
0.5	2.157688E-05
0.6	1.376867E-05
0.7	2.276897E-05
0.8	1.284480E-05
0.9	2.232194E-05

Table 3: Numerical Results for Example 3

t	Absolute Error by Proposed Method
0.1	3.695488E-06
0.2	5.424023E-06
0.3	3.099442E-06
0.4	6.198883E-06
0.5	3.218651E-06
0.6	4.947186E-06
0.7	4.649162E-06
0.8	3.665686E-06
0.9	5.483627E-06

Example 4

Consider the following linear boundary value problem

$$v^{(4)} - v = -4(2t\cos t + 3\sin t), \qquad 0 < t < 1 \tag{45}$$

Subject to $v(0)=0,\ v(1)=0,\ v''(0)=0, v''(1)=2\sin 1 + 4\cos 1$.

The exact solution for the above problem is $v = (t^{2}-1)\sin t$.

Dividing the domain [0, 1] into 10 equal subintervals, the numerical results obtained for this problem are presented in Table 4.

Example 5

Consider the following linear boundary value problem

$$v^{(4)} - 3601v'' + 3600v = -1 + 1800t^2, \qquad 0 < t < 1 \tag{46}$$

Subject to $v(0) = 1, \quad v(1) = 1.5 + \sinh(1), \; v'(0) - v(0) = 0, \; v'(1) - v(1) = -0.5 + \cosh(1) - \sinh(1)$.

The exact solution for the above problem is $v = 1 + 0.5t^2 + \sinh(t)$.

Dividing the domain [0, 1] into 10 equal subintervals, the numerical results obtained for this problem are presented in Table 5.

Table 4: Numerical Results for Example 4

t	Absolute Error by Proposed Method
0.1	2.697110E-06
0.2	4.470348E-07
0.3	4.142523E-06
0.4	3.278255E-07
0.5	4.440546E-06
0.6	3.278255E-07
0.7	4.500151E-06
0.8	3.576279E-07
0.9	4.261732E-06

Table 5: Numerical Results for Example 5

t	Absolute Error by Proposed Method
0.1	2.384186E-06
0.2	2.264977E-06
0.3	4.768372E-06
0.4	2.741814E-06
0.5	2.980232E-06
0.6	4.768372E-07
0.7	7.152557E-07
0.8	1.668930E-06
0.9	2.384186E-07

Example 6

Consider the following nonlinear boundary value problem

$$v^{(4)} - 6e^{-4v} = -12(1+t)^{-4}, \quad 0 < t < 1 \tag{47}$$

Subject to $v(0) = 0, v(1) = \ln 2, v'(0) = 1, v'(1) = 0.5$.

The exact solution for the above problem is $v = ln(1+t)$.

By using quasilinearization technique [22], the nonlinear boundary value problem (47) is converted into a sequence of linear boundary value problems as

$$v_{(n+1)}^{(4)}+[24e^{-4v_{(n)}}]v_{(n+1)}=-12(1+t)^{-4}+e^{-4v_{(n)}}[6+24v_{(n)}],\quad n=0,1,2,3,... \tag{48}$$

Subject to $v_{(n+1)}(0)=0, v_{(n+1)}(1)=\ln 2, v'_{(n+1)}(0)=1, v'_{(n+1)}(1)=0.5$.

Here $v_{(n+1)}$ is the $(n+1)^{th}$ approximation for v. The domain [0, 1] is divided into 10 equal subintervals and the proposed method is applied to the sequence of linear problems (48). The obtained numerical results for this problem are presented in Table 6.

Table 6: Numerical Results for Example 6

t	Absolute Error by proposed Method
0.1	6.929040E-07
0.2	2.384186E-07
0.3	5.066395E-07
0.4	0.000000E+00
0.5	3.874302E-07
0.6	2.384186E-07
0.7	3.576279E-07
0.8	1.788139E-07
0.9	0.000000E+00

Example 7

Consider the following nonlinear boundary value problem

$$v^{(4)}-v'^2+vv'''=-4t^2+e^t(1+t^2-4t),\quad 0<t<1 \tag{49}$$

Subject to $v(0)=1, v(1)=1+e, v'(0)=1, v'(1)=2+e$.

The exact solution for the above problem is $v=t^2+e^t$.

By using quasilinearization technique [22], the nonlinear boundary value problem (49) is converted into a sequence of linear boundary value problems as

$$v_{(n+1)}^{(4)}+v_{(n)}v_{(n+1)}'''-2v_{(n)}'v_{(n+1)}'+v_{(n)}'''v_{(n+1)}=-4t^2+e^t(1+t^2-4t)-\left(v_{(n)}'\right)^2+v_{(n)}v_{(n)}''',\ \text{n=0,1,2,,...} \tag{50}$$

Subject to $v_{(n+1)}(0)=1, v_{(n+1)}(1)=1+e, v'_{(n+1)}(0)=1, v'_{(n+1)}(1)=2+e$.

Here $v_{(n+1)}$ is the $(n+1)^{th}$ approximation for *v*. The domain [0, 1] is divided into 10 equal subintervals and the proposed method is applied to the sequence of linear problems (50). The obtained numerical results for this problem are presented in Table 7.

Table 7: Numerical Results for Example 7

t	Absolute Error by Proposed Method
0.1	9.536743E-07
0.2	2.980232E-06
0.3	2.741814E-06
0.4	5.602837E-06
0.5	3.933907E-06
0.6	6.675720E-06
0.7	2.145767E-06
0.8	2.384186E-06
0.9	2.145767E-06

Example 8

Consider the following nonlinear boundary value problem

$$v^{(4)} = \sin t + \sin^2 t - [v'']^2, \qquad 0 < t < 1 \tag{51}$$

Subject to $v(0) = 0,\ v(1) = \sin 1, v''(0) = 0,\ v''(1) = -\sin 1.$

The exact solution for the above problem is $v = \sin t$.

By using quasilinearization technique [22], the nonlinear boundary value problem (51) is converted into a sequence of linear boundary value problems as

$$v^{(4)}_{(n+1)} + [2v''_{(n)}]v''_{(n+1)} = \sin t + \sin^2 t + [v''_{(n)}]^2, \quad n = 0,1,2,3,... \tag{52}$$

Subject to $v_{(n+1)}(0) = 0,\ \ v_{(n+1)}(1) = \sin 1, v''_{(n+1)}(0) = 0,\ v''_{(n+1)}(1) = -\sin 1.$

Here $v_{(n+1)}$ is the $(n+1)^{th}$ approximation for *v*. The domain [0, 1] is divided into 10 equal subintervals and the proposed method is applied to the sequence of linear problems (52). Numerical results for this problem are presented in Table 8.

Table 8: Numerical Results for Example 8

t	Absolute Error by Proposed Method
0.1	8.456409E-06
0.2	1.586974E-05
0.3	2.205372E-05
0.4	2.574921E-05
0.5	2.697110E-05
0.6	2.521276E-05
0.7	2.157688E-05
0.8	1.531839E-05
0.9	7.987022E-06

Example 9

Consider the following nonlinear boundary value problem

$$v^{(4)} - v^2 = e^t - t^4 - e^{2t} - 2t^2 e^t, \quad 0 < t < 1 \tag{53}$$

Subject to $v(0) = 1, v(1) = 1 + e, v'(0) - v(0) = 0, v'(1) - v(1) = 1$.

The exact solution for the above problem is $v = t^2 + e^t$.

By using quasilinearization technique [22], the nonlinear boundary value problem (53) is converted into a sequence of linear boundary value problems as

$$v_{(n+1)}^{(4)} - 2v_{(n)}v_{(n+1)} = e^t - t^4 - e^{2t} - 2t^2 e^t - v^2_{(n)}, \quad \text{n=0, 1, 2, 3,…} \tag{54}$$

Subject to $v_{(n+1)}(0) = 1, v_{(n+1)}(1) = 1 + e, v'_{(n+1)}(0) - v_{(n+1)}(0) = 0, v'_{(n+1)}(1) - v_{(n+1)}(1) = 1$.

Here $v_{(n+1)}$ is the $(n+1)^{th}$ approximation for *v*. The domain [0, 1] is divided into 10 equal subintervals and the proposed method is applied to the sequence of linear problems (54). The obtained numerical results for this problem are presented in Table 9.

Table 9: Numerical results for Example 9

t	Absolute Error by Proposed Method
0.1	1.668930E-06
0.2	1.192093E-07
0.3	2.026558E-06
0.4	1.668930E-06
0.5	3.695488E-06
0.6	2.145767E-06
0.7	5.245209E-06
0.8	3.814697E-06
0.9	4.053116E-06

CONCLUSIONS

In this paper, we have solved a general fourth order two point boundary value problem with three different cases of boundary conditions by the proposed method with cubic B-splines as basis functions and quartic B-splines as weight functions. The cubic B-splines and quartic B-splines are redefined into new sets of functions which contain the equal number of functions. To test the accuracy and efficiency of the developed method, it has been tested on five linear and four nonlinear fourth order boundary value problems. It is found that the obtained results are giving a little error. The strength of the developed method lies in the easiness of its application, accuracy and efficiency.

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